Tutorial 6 for MATH 2020A (2024 Fall)

`Please check the updated tutorial 5 notes for an important correction.`

1. Give a formula $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ for the vector field $\mathbf{F}$ defined in the $xy-$plane except the origin such that

   (a) $\mathbf{F}(x, y)$ points toward the origin with magnitude inversely proportional to the square of the distance from (x, y) to the origin.

   (b) $\mathbf{F}(x, y)$ is a unit vector pointing toward the origin.

> **Solution:** (a)$F(x, y) = (\frac{-x}{(x^2+y^2)^{\frac{3}{2}}}, \frac{-y}{(x^2+y^2)^{\frac{3}{2}}})$
> (b)$F(x, y) = (\frac{-x}{(x^2+y^2)^{\frac{1}{2}}}, \frac{-y}{(x^2+y^2)^{\frac{1}{2}}})$

2. Let C be the space curve $\mathbf{r}(t) = t\mathbf{i} - \mathbf{j} + t^2\mathbf{k}$, $0 \leq t \leq 1$, evaluate the following integrals: (a) $\int_C (x + y - z)\, \mathrm{d}x$; (b) $\int_C (x + y - z)\, \mathrm{d}y$; (c) $\int_C (x + y - z)\, \mathrm{d}z$.

> **Solution:** (a)$\frac{-5}{6}$; (b)0; (c)$\frac{-5}{6}$.

3. Let $C_1 : \mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j}$, $0 \leq t \leq 2$ and $C_2 : \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$, $0 \leq t \leq 2$ be two plane curves.

   (a) Draw C_1 and C_2 in the $xy-$plane.

   (b) Find the flow of the vector field $\mathbf{F}(x, y) = y^2\mathbf{i} + 2xy\mathbf{j}$ along C_1 and C_2.

> **Solution:** along C_1: 32; along C_2: 32. (Not a coincidence, note $\mathbf{F} = \nabla(xy^2)$, a conservative vector field!)

4. Suppose that $f(t)$ is a differentiable and positive function for $a \leq t \leq b$. Let C be the path $\mathbf{r}(t) = t\mathbf{i} + f(t)\mathbf{j}$, $a \leq t \leq b$.

   (a) Draw the path C in the $xy-$plane.

   (b) Let the region $D \subset \mathbb{R}^2$ be the region bounded by the $x-$axis, the curve C, and the lines $x = a$, $x = b$, calculate $\iint_D 1\, \mathrm{d}A$.

   (c) Let the vector field $\mathbf{F}$ be given by $\mathbf{F}(x, y) = y\mathbf{i}$, show that

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_D 1\, \mathrm{d}A.$$

Solution: (b)$\int_a^b f(t)\,\mathrm{d}t$;
(c)$LHS = \int_C \mathbf{F} \cdot\, \mathrm{d}\mathbf{r} = \int_a^b f(t)\,\mathrm{d}t = RHS$.
Question (c) is actually a special case of Green's theorem.